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On Sieved Orthogonal Polynomials. IV. Generating Functions*

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We study the structure of generating functions of sieved polynomials and their numerators. Examples and applications are mentioned.

1. INTRODUCTION

The sieved ultraspherical polynomials were introduced in [1]. The sieved ultraspherical polynomials of the first kind $c_n^\lambda(x; k)$ are generated by

$$2xc_n^\lambda(x; k) = c_{n+1}^\lambda(x; k) + c_{n-1}^\lambda(x; k), \qquad n \neq mk,$$
$$2x(m+\lambda)\, c_{mk}^\lambda(x; k) = (m+2\lambda)\, c_{mk+1}^\lambda(x; k) + mc_{mk-1}^\lambda(x; k), \qquad m > 1, \tag{1.1}$$

with $c_0^\lambda(x; k) = 1$, $c_1^\lambda(x; k) = x$, and the sieved ultraspherical polynomials of the second kind $B_\eta^\lambda(x; k)$ are generated by

$$2xB_n^\lambda(x; k) = B_{n+1}^\lambda(\eta; k) + B_{n-1}^\lambda(x; k), \qquad n+1 \neq mk$$
$$2x(m+\lambda)\, B_{mk-1}^\lambda(x; k) = mB_{mk}^\lambda(x; k) + (m+2\lambda)\, B_{mk-2}^\lambda(x; k), \qquad m > 1, \tag{1.2}$$

where $B_0^\lambda(x; k) = 1$ and $B_1^\lambda(x; k) = 2x$ if $k > 1$; $B_1^\lambda(x; 1) = 2(\lambda+1)\, x$. These polynomials are orthogonal; for their orthogonality relation see [1]. In [8] we introduced a generalization of the c_n^λ's and B_n^λ's analoguous to the

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generalizations of the ultraspherical polynomials due to Pollaczek [12] and Szegö [15]. The weight functions for our polynomials were computed in [8]. The work [8] was part I of a series of papers on sieved orthogonal polynomials. Part II, [7], dealt with sieved random walk polynomials while part III, [9], treated sieved polynomials that are orthogonal on several intervals.

The purpose of this note is to study the structure of generating functions of general sieved polynomials of the second kind. We have been unsuccessful in obtaining analoguous results for sieved polynomials of the first kind. Generating functions can be used to obtain asymptotic expansions for the polynomials, say $\{p_n(x)\}$, for large n and fixed x in the complex x plane. The asymptotic behavior of the polynomials usually determines the spectral measure(s) of the associated Jacobi matrix. That spectral measure is the measure the polynomials are orthogonal with respect to, see Askey and Ismail [3], Nevai [11]. Recovering the spectral measure from the asymptotic behavior of $p_n(x)$ is, in a way, a discrete analogue of the inverse scattering problem, Case [5] and, Case and Kac [6].

The polynomials that we shall study are generated by

$$2xa_np_n(x) = b_np_{n+1}(x) + c_np_n(x) + d_np_{n-1}(x), \quad n \geq 0 \tag{1.3}$$

with

$$p_0(x) = 1, \qquad p_{-1}(x) = 0, \tag{1.4}$$

$$a_n = b_n = d_n = 1, \qquad c_n = 0 \qquad \text{if} \quad k \nmid n+1, \tag{1.5}$$

and

$$a_{mk-1},\ b_{mk-1},\ c_{mk-1}, \text{ and } d_{mk-1} \quad \text{are polynomials in } m. \tag{1.6}$$

The integer k is assumed to be at least 2. In Section 2 we first investigate the structure of the generating functions

$$\sum_{n=0}^{\infty} p_n(x)\, t^n \quad \text{and} \quad \sum_{n=0}^{\infty} p_{nk+l}(x)\, t^n, \quad l = 0, 1, \dots, k-1.$$

These structure theorems are Theorems 2.1 and 2.2. The polynomials $\{p_n(x)\}$ defined by (1.3) and (1.4) are called the denominator polynomials because they are the denominators of the associated continued fraction. The numerator polynomials $\{p_n^*(x)\}$ are the solutions of (1.3) that satisfy the initial conditions

$$p_0^*(x) = 0, \qquad p_1^*(x) = 2a_0/b_0. \tag{1.7}$$

In view of (1.4) and (1.5) we have

$$p_0(x)=1, \quad p_1(x)=2x, \quad p_0^*(x)=0, \quad p_1^*(x)=2. \tag{1.8}$$

In Section 2 we also investigate the structure of the generating functions

$$\sum_{n=1}^{\infty} p_n^*(x)\, t^n \quad \text{and} \quad \sum_{n=0}^{\infty} p_{nk+l}^*(x)\, t^n, \quad l=0, 1,..., k-1.$$

Our results are stated as Theorems 2.3 and 2.4.

A classical theorem of Markov (Szegö [16]) asserts that if the polynomials $\{p_n(x)\}$ are orthogonal with respect to a positive measure $d\mu$, that is,

$$\int_{-\infty}^{\infty} p_n(x)\, p_m(x)\, d\mu(s) = \lambda_n \delta_{m,n}, \quad \lambda_0 = 1, \tag{1.9}$$

and $d\mu$ has a compact support then

$$\lim_{n\to\infty} p_n^*(z)/p_n(z) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{z-t}. \tag{1.10}$$

The left-hand side of (1.9) is the continued fraction associated with the polynomials $\{p_n(x)\}$. So, the asymptotic behavior of $p_n(z)$ and $p_n^*(z)$ determines the left-hand side of (1.9) and μ can then be computed from the Perron–Stieltjes inversion formula

$$F(z)=\int_{-\infty}^{\infty} \frac{d\psi(t)}{z-t} \qquad \text{iff} \quad \psi(t_2)-\psi(t_1) = \lim_{\varepsilon\to 0+} \int_{t_1}^{t_2} \frac{F(t-i\varepsilon)-F(t+i\varepsilon)}{2\pi i}\, dt, \tag{1.11}$$

which holds when the support of $d\psi$ is contained in a half line.

Generating functions have at least two important uses. They usually lead to explicit formulas for the polynomials under consideration. Second, Darboux's asymptotic method, Olver [12, Sect. 8.9], can frequently be applied to generating functions in order to determine the asymptotic behavior of the polynomials involved. Examples of this procedure are in Olver [12], Szegö [16], and Askey and Ismail [3].

Section 3 contains a sieved analogue of Roger's q-Hermite polynomials. Rogers used his q-Hermite and q-ultraspherical polynomials to prove the Rogers–Ramanujan identities, see [2] for references and details. Ismail and Stanton [10] used the linearization of products of Roger's q-hermite polynomials to prove the orthogonality relation of Askey and Wilson's ${}_4\phi_3$ polynomials. The ${}_4\phi_3$ polynomials are q-analogues of the 6-j symbols (Askey and Wilson [4]).

The results of this work have been used in [9]. Let $\{p_n(x)\}$ be a sequence of polynomials such that

$$p_0(x) = 1, \qquad p_1(x) = A_0 x + B_0 \tag{1.12}$$

and let $p_n(x)$ be of precise degree n. It is well known that $\{p_n(x)\}$ is orthogonal if and only if it satisfies a three term recurrence relation

$$p_{n+1}(x) = (A_n x + B_n)\, p_n(x) - C_n p_{n-1}(x), \qquad n > 0, \tag{1.13}$$

and a positivity condition

$$A_n A_{n-1} C_n > 0, \qquad n = 1, 2, 3, \ldots. \tag{1.14}$$

When (1.13) and (1.14) hold the orthogonality relation will be

$$\int_{-\infty}^{\infty} p_n(x)\, p_m(x)\, d\mu(x) = \lambda_n \delta_{m,n}, \tag{1.15}$$

with

$$\lambda_n = A_0 C_1 C_2 \cdots C_n / A_n, \qquad n > 0,\ \lambda_0 = 1. \tag{1.16}$$

The condition $\lambda_0 = 1$ is a normalization condition. When the sequence $\{B_n/A_n\}$ and $\{C_n/(A_n A_{n-1})\}$ are bounded the support of $d\mu$ will be bounded and Markov's theorem will be applicable. This information will be used in Section 3.

2. Generating Functions

Set

$$G_l(x; t) := \sum_{n=0}^{\infty} p_{nk+l}(x)\, t^n, \qquad l = 0, 1, \ldots, k-1. \tag{2.1}$$

It is easy to see that if

$$\omega = \exp(2\pi i/k), \tag{2.2}$$

then

$$\sum_{j=0}^{k-1} \omega^{-jl} \sum_{n=0}^{\infty} (t\omega^j)^n\, h_n = k \sum_{n=0}^{\infty} h_{nk+l} t^{kn+l} \tag{2.3}$$

for $l = 0, 1, \ldots, k-1$. Now multiply (1.3) by $t^n \omega^{jn}$ and add the resulting identities for $n = 0, 1, \ldots,$ then multiply the resulting series by ω^{-jl} and add for

$j=0, 1,..., k-1$. This and the observation (2.3) give, after replacing t^k by t, the following identities

$$\sum_{n=0}^{\infty} (2xa_{nk+l} - c_{nk+l})\, p_{nk+l}(x)\, t^n = \sum_{n=0}^{\infty} b_{nk+l} p_{nk+l+1}(x)\, t^n + \sum_{n=0}^{\infty} d_{nk+l} p_{nk+l-1}(x)\, t^n, \tag{2.4}$$

where $l=0, 1,..., k-1$. In terms of the generating functions $G_l(x, t)$, (2.4) becomes

$$\begin{aligned} 2xG_0(x, t) &= G_1(x, t) + G_{k-1}(x, t), \\ 2xG_l(x, t) &= G_{l+1}(x,t) + G_{l-1}(x, t), \quad l=1, 2,..., k-2, \end{aligned} \tag{2.5}$$

$$\sum_{n=0}^{\infty} (2xa_{nk+k-1} - c_{nk+k-1})\, p_{nk+k-1}(x)\, t^n = \sum_{n=0}^{\infty} b_{nk+k-1} p_{nk+k}(x)\, t^n + \sum_{n=0}^{\infty} d_{nk+k-1} p_{nk+k-2}(x)\, t^n. \tag{2.6}$$

Observe that (2.6) is a differential equation involving $G_0(x, t)$, $G_{k-1}(x, t)$, $G_{k-2}(x, t)$, and their derivatives (with respect to t). We now prove

THEOREM 2.1. *The generating functions $G_l(x, t)$ of (2.1) are of the form*

$$G_l(x, t) = F(x, t)\{U_l(x) + tU_{k-l-2}(x)\}, \qquad l=0, 1,..., k-1, \tag{2.7}$$

where $F(x, t)$ is a power series depending on x and t but does not depend on l. The U's are the Chebychev polynomials

$$U_n(x) = \frac{\sin(n+1)\,\theta}{\sin\theta}, \qquad x=\cos\theta, n=-2, -1, 0, 1,.... \tag{2.8}$$

Proof. The recursion (1.3) and the initial conditions (1.4) define the polynomials $\{p_n(x)\}$ uniquely, hence the G_l's are uniquely defined by (2.1). The recurrence relation

$$2xU_n(x) = U_{n+1}(x) + U_{n-1}(x), \qquad n=0, 1,..., \tag{2.9}$$

and $U_0(x)=1$, $U_{-1}(x)=0$, $U_{-2}(x)=-U_0(x)=-1$, can be used to show that the right-hand sides of (2.7) satisfy (2.5) for arbitrary $F(x, t)$. Finally the substitution of the right-hand sides of (2.7) for $l=0, k-2, k-1$ gives a differential equation (in t) with polynomial coefficients whose solution $F(x, t)$ obviously depends only on x, t and k. The integration constants involved in determining $F(x, t)$ are functions of x and k and can be determined uniquely from the first few p's in (2.1) and (2.7). This completes the proof.

We now derive a full generating function. Let

$$G(x, t) = \sum_{l=0}^{k-1} t^l G_l(x, t^k) = \sum_{0}^{\infty} p_n(x)\, t^n \tag{2.10}$$

THEOREM 2.2. *We have*

$$\sum_{n=0}^{\infty} p_n(\cos\theta)\, t^n = \left(\frac{1 - 2t^k \cos k\theta + t^{2k}}{1 - 2t\cos\theta + t^2}\right) F(\cos\theta, t^k). \tag{2.11}$$

Proof. The relationship (2.7) and (2.9) imply

$$G(x, t) = F(x, t^k) \sum_{l=0}^{k-1} \{t^l U_l(x) + t^{k+l} U_{k-l-2}(x)\}. \tag{2.12}$$

It is easy to obtain

$$\sum_{l=0}^{k-1} t^l U_l(x) = \{1 - t^k U_k(x) + t^{k+1} U_{k-1}(x)\}/(1 - 2xt + t^2),$$

and

$$\sum_{l=0}^{k-1} t^{k+l} U_{k-l-2}(x) = \{t^{2k} + t^k U_{k-2}(x) - t^{k+1} U_{k-1}(x)\}/(1 - 2xt + t^2),$$

from (2.8) and standard trigonometric identities. The result now follows from (2.12) and

$$U_k(\cos\theta) - U_{k-2}(\cos\theta) = 2\cos k\theta, \tag{2.13}$$

and the proof is complete.

We illustrate the aformentioned procedure by treating the sieved polynomials $B_n^\lambda(x; k)$ of (1.2). In this case

$$a_{nk-1} = \lambda + n, \quad b_{nk-1} = n, \quad c_{nk-1} = 0, \quad d_{nk-1} = n + 2\lambda,$$

and (2.6) becomes

$$\begin{aligned} &2x(\lambda + 1)\, G_{k-1}(x, t) + 2xt \frac{\partial}{\partial t} G_{k-1}(x, t) \\ &\quad = \frac{\partial}{\partial t} G_0(x, t) + (2\lambda + 1)\, G_{k-2}(x, t) + t \frac{\partial}{\partial t} G_{k-2}(x, t). \end{aligned} \tag{2.14}$$

Using (2.7), (2.9), and (2.13) we get

$$2(\lambda+1)\{\cos(k\theta)-t\}\, F(\cos\theta, t)=\{1-2t\cos(k\theta)+t^2\}\,\frac{\partial}{\partial t} F(\cos\theta, t).$$

Therefore

$$F(\cos\theta, t)=\{1-2t\cos(k\theta)+t^2\}^{-\lambda-1}. \tag{2.15}$$

This and (2.11) yield

$$\sum_{n=0}^{\infty} B_n^{\lambda}(\cos\theta; k)\, t^n=(1-2t\cos\theta+t^2)^{-1}(1-2t^k\cos k\theta+t^{2k})^{-\lambda}. \tag{2.16}$$

The generating function (2.16) was derived in [1] as a limiting case of the generating function of the continuous q-ultraspherical polynomials, but the above direct proof is new. Another proof is in [9].

We now study generating functions for $\{p_n^*(x)\}$. Let

$$G_l^*(x, t) := \sum_{n=0}^{\infty} p_{nk+l}(x)\, t^n, \quad l=0, 1,..., k-1. \tag{2.17}$$

It is easy to see that

$$U_n^*(x)=2U_{n-1}(x), \tag{2.18}$$

and that the generating function

$$U_l^*(x, t) := \sum_{n=0}^{\infty} U_{nk+l}^*(x)\, t^n \tag{2.19}$$

is given by

$$U_l^*(x, t)=2[U_{l-1}(x)+tU_{k-l-1}(x)]/[1-2tT_k(x)+t^2]. \tag{2.20}$$

Observe that

$$p_l^*(x)=U_l^*(x)=2U_{l-1}(x), \qquad l=0, 1,..., k-1. \tag{2.21}$$

The recursion (1.3) implies

$$2xp_{nk+l}^*(x)=p_{nk+l+1}^*(x)+p_{nk+l-1}^*(x), \qquad n\geqslant 0,\ l=0, 1,..., k-2, \tag{2.22}$$

which leads to

$$2xG_l^*(x, t)=G_{l+1}^*(x, t)+G_{l-1}^*(x, t), \qquad l=1, 2,..., k-2. \tag{2.23}$$

When $l=0$, (2.22) yields the relationship

$$2x\{G_0^*(x,t)-p_0^*(x)\}=G_1^*(x,t)-p_1^*(x)+tG_{k-1}^*(x,t).$$

Therefore

$$2xG_0^*(x,t)=G_1^*(x,t)+tG_{k-1}^*(x,t)-2. \tag{2.24}$$

Clearly $U_l^*(x,t)$, $l=0,1,\ldots,k-1$ also satisfies the inhomogeneous system (2.21) and (2.22), hence $G_l^*(x,t)-U_l^*(x,t)$ satisfies the corresponding homogeneous system which is identical with (2.5). This and Theorem 2.1 establish

THEOREM 2.3. *The generating functions $G_l^*(x,t)$ of* (2.15) *have the form*

$$G_l^*(x,t)=U_l^*(x,t)+\{U_l(x)+tU_{k-l-2}(x)\}\,F^*(x,t), \tag{2.25}$$

where $F^*(x,t)$ depends on x, t and k but is independent of l.

Now Theorem 2.3 and an argument similar to what we used to prove Theorem 2.2 give

THEOREM 2.4. *A generating function for the numerator polynomials is*

$$\sum_{n=0}^{\infty} p_n^*(x)\,t^n=\frac{2t}{1-2xt+t^2}+\frac{1-2t^kT_k(x)+t^{2k}}{1-2xt+t^2}\,F^*(x,t^k). \tag{2.26}$$

We now treat the numerator polynomials associated with the sieved ultraspherical polynomials. In this case the differential equation (2.14) remains valid with the G's replaced by G^*'s. Applying (2.24) we obtain after some tedious calculations

$$\begin{aligned}\{1-2tT_k(x)+t^2\}\frac{\partial}{\partial t}F^*(x,t)&+2(\lambda+1)\{t-T_k(x)\}\,F^*(x,t)\\ &=2\lambda\{e^{i(k-1)\theta}/(1-te^{ik\theta})+e^{-i(k-1)\theta}/(1-te^{-ik\theta})\},\end{aligned}$$

$x=\cos\theta$. This shows

$$F^*(x,t)=F(x,t)\{H(t,\theta)+H(t,-\theta)\} \tag{2.27}$$

where

$$H(t,\theta):=2\lambda e^{i(k-1)\theta}\int_0^t(1-ue^{ik\theta})^{\lambda-1}(1-ue^{-ik\theta})^{\lambda}\,du, \tag{2.28}$$

and $F(x,t)$ is given by (2.15).

In the above analysis $e^{\pm i\theta}=x\pm\sqrt{x^2-1}$ and it is easy to see that $|e^{-i\theta}|<|e^{i\theta}|$ if and only if x is in the complex plane cut along $[-1, 1]$. Applying Darboux's method to (2.16) and (2.26) we obtain the relationship

$$\int_{-\infty}^{\infty} \frac{d\psi(t)}{x-t}=2\left\{e^{-i\theta}-2i\lambda \sin\theta\, e^{-2ik\theta}\int_0^1 (1-ue^{-2ik\theta})^{\lambda-1}(1-u)^{\lambda}\, du\right\},$$

which can be inverted using (1.11) to obtain the distribution function ψ.

3. Sieved q-Hermite Polynomials

We now treat the case when the coefficients a_n, b_n, c_n, and d_n in (1.3) satisfy (1.5) but (1.6) is replaced by

$$a_{mk-1}, b_{mk-1}, c_{mk-1}, \text{ and } d_{mk-1} \quad \text{are polynomials in } q^m, \tag{3.1}$$

where $q\in(-1, 1)$. The results of Section 2 remain valid in this case and (2.6) becomes a q-difference equation, that is an equation involving $F(x, t)$, $F(x, qt)$,..., $F(x, q^j t)$. We illustrate the method by considering the example

$$a_{mk-1}=d_{mk-1}=1, \qquad b_{mk-1}=1-q^m, \qquad c_{mk-1}=0. \tag{3.2}$$

Recall that the case $k=1$ of this example is the continuous q-Hermite polynomials that L. J. Rogers used to prove the celebrated Rogers–Ramanujan identities. In this case (2.6) becomes

$$G_0(x, t)=G_0(x, qt)+2xtG_{k-1}(x, t)-tG_{k-2}(x, t),$$

which, inview of (2.7) gives

$$F(x, t)=\{1+qt\, U_{k-2}(x)\}\{1-2tT_k(x)+t^2\}^{-1} F(x, qt).$$

The above functional equation can be iterated to give

$$F(x, t)=(-qtU_{k-2}(x); q)_\infty/\{te^{ik\theta}; q)_\infty(te^{-ik\theta}; q)_\infty\} \tag{3.3}$$

where

$$(\sigma; q)_0=1,\ (\sigma; q)_n=\prod_{j=1}^{n}(1-\sigma q^{j-1}), \qquad n=1, 2, \ldots, \text{ or } \infty.$$

Now Theorem 2.2 implies

$$\sum_{n=0}^{\infty} H_n(x; k\,|\,q)\, t^n=\frac{(-qt^kU_{k-2}(x); q)_\infty}{(1-2tx+t^2)(qt^ke^{ik\theta}; q)_\infty(qt^ke^{-ik\theta}; q)_\infty}. \tag{3.4}$$

We denoted the polynomials under investigation by $H_n(x; k \mid q)$. When $k=1$ these polynomials reduce to $H_n(x \mid q)/(q; q)_n$. Furthermore, Theorem 2.1 and (3.3) yield

$$\sum_{n=0}^{\infty} H_{nk+l}(x; k \mid q)\, t^n = \frac{\{U_l(x) + tU_{k-l-2}(x)\}(-qtU_{k-2}(x); q)_\infty}{(te^{ik\theta}; q)_\infty (te^{-ik\theta}; q)_\infty}, \quad (3.5)$$

$l = 0, 1, ..., k-1$. Next we apply Darboux's method to (3.4) and obtain

$$H_n(x; k \mid q) \approx 2Re\left[\frac{(-qe^{ik\theta}U_{k-2}(x); q)_\infty\, e^{-in\theta}}{(1-e^{2i\theta})(q; q)_\infty (qe^{2ik\theta}; q)_\infty}\right], \quad (3.6)$$

as $n \to \infty$, holding for $x \in (-1, 1)$. The asymptotic relationship (3.6) can be simplified to

$$H_n(x; k \mid q) \approx 2\left|\frac{U_{k-1}(x)(-qe^{ik\theta}U_{k-2}(x); q)_\infty}{(q; q)_\infty (e^{2ik\theta}; q)_\infty}\right| \cos(n\theta + \varepsilon), \quad (3.7)$$

as $n \to \infty$ for some ε which depends only on θ.

We now proceed to compute the (positive) measure that our polynomials are orthogonal with respect to. In the present case the coefficients A_n, B_n, and C_n of (1.13) are given by

$$\begin{aligned} &B_n = 0,\ A_n = 2,\ C_n = 1 && \text{if} \quad k \nmid n+1 \\ &B_{mk-1} = 0,\ A_{mk-1} = 1/(1-q^m),\ C_{mk-1} = 1/1-q^m), && m = 1, ... \end{aligned} \quad (3.8)$$

The positivity condition (1.14) is obviously satisfied and (1.16) becomes

$$\lambda_{mk+k-1} = 1/(q; q)_{m+1+l} = 1/(q; q)_m, \qquad l = 0, 1, ..., k-2, \quad (3.9)$$

$m = 0, 1, ...$ Therefore there exists a nondecreasing function $\sigma(x)$ of bounded variation on $(-\infty, \infty)$ such that

$$\int_{-\infty}^{\infty} H_n(x; k \mid q)\, H_m(x; k \mid q)\, d\sigma(x) = \lambda_n \delta_{m,n}, \quad (3.10)$$

with λ_n as in (3.9). A theorem of Nevai [11] asserts that if the series

$$\sum_{n=1}^{\infty} \{|B_n/A_n| + |(C_n/A_nA_{n-1})^{1/2} - \alpha|\} \quad (3.11)$$

converges then

$$d\sigma(x) = \sigma'(x)\, dx + d\sigma_j(x), \quad (3.12)$$

where $\sigma'(x)$ is continuous and positive in $(-\alpha, \alpha)$, supp $\sigma'(x) = [\alpha, \alpha]$ and $\sigma_j(x)$ is a step function (with possibly infinitely many jumps) constant in $(-\alpha, \alpha)$. Furthermore the limiting relation

$$\limsup_n \sigma'(x) \sqrt{\alpha^2 - x^2}\, p_n^2(x)/\lambda_n = 2/\pi, \tag{3.13}$$

holds. The above version of Nevai's theorem follows from Corollary 36 (p. 141) and Theorem 40 (p. 143) in Nevai's memoir [11].

In the case of the polynomials $\{H_n(x; k \mid q)\}$, $\alpha = 1$ and the convergence of the series in (3.11) follows from (3.8). Note that $\lambda_n \approx 1/(q; q)_\infty$ follows from (3.9). Thus (3.13) holds and we have

$$\sigma'(\cos\theta) = \frac{(q; q)_\infty \sin\theta (qe^{ik\theta}; q)_\infty (qe^{-ik\theta}; q)_\infty}{2\pi(-qe^{ik\theta} U_{k-2}(x); q)_\infty (-qe^{-ik\theta} U_{k-2}(x); q)_\infty}. \tag{3.14}$$

In the present case the support of $d\sigma(x)$ is clearly bounded. Recall that when $d\sigma$ has compace support the associated moment problem is determined and Corollary 2.6 (pp. 45–46) in Shohat and Tamarkin [14] is applicable, that is $\sigma(x)$ has a jump at $x = \xi$ if and only if $\sum_{n=0}^{\infty} p_n^2(\xi)/\lambda_n$ diverges. In the case of $\{H_n(x; k \mid q)\}$ it is easy to see that

$$H_n(1; k \mid q) = (-1)^n H_n(-1; k \mid q) = An + B + o(1),$$

where A, B depend on q and A and B do not vanish simultaneously, thus $\sum H_n^2(\pm 1; k \mid q)/\lambda_n$ diverges since $\lambda_n \approx 1/(q; q)_\infty$. We now apply the same argument to any $x \notin [-1, 1]$. Since the polynomials are symmetric the measure $d\sigma$ must be symmetric in x, so there is no loss of generality in assuming $x > 1$. For such x define θ by $x = \cos\theta$, $|e^{-i\theta}| < |e^{i\theta}|$, Im $\theta > 0$. In this case applying Darboux's method to (3.4) establishes

$$H_n(x; k) \mid q) \approx \frac{(-qe^{-ik\theta} U_{k-2}(x); q)_\infty e^{in\theta}}{(q; q)_\infty (qe^{-2ik\theta}; q)_\infty (1 - e^{-2i\theta})}, \tag{3.15}$$

as $n \to \infty$, $x = \cos\theta > 1$, which will imply the divergence of $\sum H_n^2(x; k \mid q)/\lambda_n$ when

$$(-qe^{-ik\theta} U_{k-2}(x); q)_\infty \neq 0. \tag{3.16}$$

It is clear that (3.16) is valid when $1 > q \geqslant 0$. Now let $-1 < q < 0$ but $q(1-k) < 1$. The first term in the infinite product on the left side of (3.16) is

$$1 + qe^{-ik\theta} U_{k-2}(x) = 1 + qw(1 - w^{k-1})/(1 - w), \quad w := e^{-2i\theta}. \tag{3.17}$$

Since $w \in (0, 1)$ the function $w(1 - w^{k-1})/(1 - w)$, being $\sum_1^{k-1} w^j$, is strictly monotone increasing and attains every value in $(0, k-1)$. This and (3.17) show that $1+qe^{-ik\theta}U_{k-2}(x)>0$, hence all the terms in the infinite product in (3.16) are positive when $0<q(1-k)<1$. We shall not treat the cases $q<0$, $q(1-k)>1$ but it seems that in those cases $\sigma(x)$ will have one jump.

We now record the orthogonality relation of the polynomials $\{H_n(x; k \mid q)\}$. The above considerations and (3.9) establish the orthogonality relation

$$\int_{-1}^{1} H_n(x; k \mid q)\, H_n(x; k \mid q)\, \sigma'(x)\, dx = \lambda_n \sigma_{m,n}, \tag{3.18}$$

where $\sigma'(x)$ is given by (3.14) and λ_n is as in (3.9).

We now study the numerator polynomials and compute the continued fraction. The q-difference equation satisfied by $F^*(x, t)$ is

$$\{1-2tT_k(x)+t^2\}\, F^*(x, t) = \{1+qtU_{k-2}(x)\}\, F^*(x, qt) - \frac{2qtU_{k-1}(x)}{1-2qtT_k(x)+q^2t^2}$$

whose solution, subject to $F^*(x, 0)=0$ is

$$F^*(x, t) = -2qtU_{k-1}(x) \sum_{n=0}^{\infty} \frac{q^n(-qtU_{k-2}(x); q)_n}{(qte^{ik\theta}; q)_{n+1}(qte^{-ik\theta}; q)_{n+1}}. \tag{3.19}$$

Theorem 2.4 and (3.19) yield the generating function

$$\sum_{n=0}^{\infty} H_n^*(x; k \mid q)\, t^n = \frac{2t}{1-2xt+t^2} - \frac{2qt^kU_{k-1}(x)}{1-2xt+t^2} \times \sum_{n=0}^{\infty} \frac{q^n(-qt^kU_{k-2}(x); q)_n}{(qt^ke^{ik\theta}; q)_{n+1}(qt^ke^{-ik\theta}; q)_{n+1}}. \tag{3.20}$$

For x in the complex plane cut along $[-1, 1]$ the sign of the radical is determined by $|x-\sqrt{x^2-1}|<|x+\sqrt{x^2-1}|$ so if $x=\cos\theta$ then $|e^{-i\theta}|<|e^{i\theta}|$. Applying Darboux's method to (3.20) and (3.4) we get

$$\lim_{n\to\infty} H_n^*(x; k \mid q)/H_n(x; k \mid q) = X(x), \tag{3.21}$$

where

$$X(x) = 2\frac{(q; q)_\infty (qe^{-2ik\theta}; q)_\infty}{(-qe^{-ik\theta}U_{k-2}(x); q)_\infty} \times \left[e^{-i\theta} - qe^{-ik\theta}U_{k-1}(x) \sum_{m=0}^{\infty} \frac{q^m(qe^{-ik\theta}U_{k-2}(x); q)_m}{(qe^{-2ik\theta}; q)_{m+1}(q; q)_{m+1}} \right]. \tag{3.22}$$

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